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ALGORITHM OF STIFFNESS AND STABILITY ANALYSIS OF BAR STRUCTURES IN A GEOMETRICALLY NONLINEAR FORMULATION

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Finite-Element Algorithm. In most papers a linear approach has been used to compute truss and frame structures. We propose an approach in which a single effective algorithm is used to study both the geometrically nonlinear subcritical state and the stability of spatial truss and frame structures.

Let us consider an arbitrary spatial frame structure with rigid nodes under arbitrary load. As a finite element, we take a rectilinear bar of constant cross-section with the x axis coinciding with the axial line and the y and z axes directed along the central axes of the bar's cross-section (Fig. 1).

Assuming the smallness of elastic strains and rotation angles, the kinematic relations for bars have the form [1]

$$\varepsilon = u_s + \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2, \quad \chi_1 = \omega_{1s}, \quad \chi_2 = \omega_{2s}, \quad \chi_3 = \varphi_s, \quad \omega_1 = -w_s, \quad \omega_2 = -v_s, \tag{1}$$

where u, v, and w are displacements along the x, y, and z axes; ω_1, ω_2, and φ are the angles of small rotations about the y, z, and x axes; ε is the axial strain; χ_1, χ_2, and χ_3 are changes in curvatures and torsions; and subscript s denotes differentiation with respect to s. The statistical relations are written as

$$T = EF\varepsilon, \quad M_1 = EJ_1\chi_1, \quad M_2 = EJ_2\chi_2, \quad M_3 = EJ_k\chi_3, \quad G = E/[2(1+\nu)]$$

(J_1, J_2, J_k are the inertia moments of the cross-section of the bar under bending curvature and torsion; E is Young's modulus; and ν is Poisson's ratio).

To approximate the displacements in the finite element we choose a linear polynomial for displacements u and a cubic one for displacements v and w:

$$\begin{gathered} u = u_0E_1(s) + u_1E_2(s), \quad v = v_0E_3(s) + v_1E_4(s) + \omega_{20}E_5(s) + \omega_{21}E_6(s), \\ w = w_0E_3(s) + w_1E_4(s) - \omega_{10}E_5(s) - \omega_{11}E_6(s). \end{gathered} \tag{2}$$

Here, E_1, E_2, E_3, E_4, E_5, and E_6 are Hermitian polynomials:

$$\begin{gathered} E_1 = 1 - s/L, \quad E_2 = s/L, \quad E_3 = 1 - 3s^2/L^2 + 2s^3/L^3, \\ E_4 = 3s^2/L^2 - 2s^3/L^3, \quad E_5 = s - 2s^2/L + s^3/L^2, \quad E_6 = -s^2/L^2 + s^3/L^2; \end{gathered} \tag{3}$$

u_0, v_0, ω_0, ω_{10}, ω_{20} u_1, v_1, w_1, ω_{11}, and ω_{21} are displacements and rotations at nodes 0 and 1 in the finite element; and L is the length of the element.

The approximation of rotations ω_1 and ω_2 follows from relations (1) and (2), and for torsion φ we use the linear approximation $\varphi = \varphi_0E_1(s) + \varphi_1E_2(s)$ (φ_0 and φ_1 are the nodal values of torsion angles).

For simplicity we divide T and ε into linear and nonlinear constituents:

$$\varepsilon = \varepsilon_l + \varepsilon_n, \quad T_l = EF\varepsilon_l, \quad T_n = EF\varepsilon_n, \quad \varepsilon_l = u_s, \quad \varepsilon_n = \frac{1}{2}(\omega_1^2 + \omega_2^2).$$

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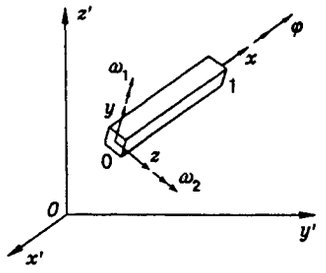

Fig. 1

The potential energy of the bar is given by

$$W_e = \frac{1}{2}\int_0^L (T\varepsilon + M_1\chi_1 + M_2\chi_2 + M_3\chi_3)\,ds = \frac{1}{2}\int_0^L (T_l\varepsilon_l + M_1\chi_1 + M_2\chi_2 + M_3\chi_3 + T_l\varepsilon_n + T_n\varepsilon_l + T_n\varepsilon_n)\,ds.$$

The nonlinear deformation and stability of the bars will be studied by the Newton method and the energy criterion of stability. To this end, we consider the first δW_e and second $\delta^2 W_e$ variations of the strain energy:

$$\delta W_e = \int_0^L (T_l\delta\varepsilon_l + M_1\delta\chi_1 + M_2\delta\chi_2 + M_3\delta\chi_3 + T_l\delta\varepsilon_n + T_n\delta\varepsilon_l + T_n\delta\varepsilon_n)\,ds,$$

$$\delta^2 W_e = \int_0^L (\delta\varepsilon_l EF\delta\varepsilon_l + \delta\chi_1 EJ_1\delta\chi_1 + \delta\chi_2 EJ_2\delta\chi_2 + \delta\chi_3 EG\delta\chi_3$$

$$+ \delta\varepsilon_l EF\delta\varepsilon_n + \delta\varepsilon_n EF\delta\varepsilon_l + \delta\varepsilon_n EF\delta\varepsilon_n + T_l\delta^2\varepsilon_n + T_n\delta^2\varepsilon_n)\,ds,$$

$$\delta\varepsilon_n = w_s\delta w_s + v_s\delta v_s, \quad \delta^2\varepsilon_n = \delta w_s\delta w_s + \delta v_s\delta v_s.$$

We take the displacements and the angles of rotation and torsion as nodal unknowns of the bar finite element and introduce the vector of the nodal unknowns

$$\mathbf{u}_e^t = \{u_0, v_0, w_0, \omega_{10}, \omega_{20}, u_1, v_1, w_1, \omega_{11}, \omega_{21}\} \tag{4}$$

(superscript t denotes transposition).

The equation of the Newton method is now written as

$$\mathbf{H}_e(\mathbf{u}_e)\,\delta\mathbf{u}_e = \mathbf{P}_e - \mathbf{G}_e(\mathbf{u}_e), \tag{5}$$

where $\mathbf{P}_e$ is the vector of external forces and moments applied to the element; the Hessian matrix $\mathbf{H}_e$ and the strain-energy gradient $\mathbf{G}_e$ are determined from the relations

$$\delta^2 W_e = \delta\mathbf{u}_e\mathbf{H}_e(\mathbf{u}_e)\delta\mathbf{u}_e, \quad \delta W_e = \mathbf{G}_e(\mathbf{u}_e)\delta\mathbf{u}_e. \tag{6}$$

Taking into account expressions (2)–(6), the Hessian matrix $\mathbf{H}_e$ and the gradient $\mathbf{G}_e$ are represented as

$$\mathbf{H}_e = \mathbf{K}_l + EF\int_0^L \left\{w_s\mathbf{K}_{11} + v_s\mathbf{K}_{12} + \left(u_s + \frac{1}{2}v_s^2 + \frac{3}{2}w_s^2\right)\mathbf{K}_{21} + v_s w_s\mathbf{K}_{22} + \left(u_s + \frac{3}{2}v_s^2 + \frac{1}{2}w_s^2\right)\mathbf{K}_{23}\right\} ds; \tag{7}$$

$$\mathbf{G}_e = \mathbf{u}_l^t\mathbf{K}_l + EF\int_0^L \left\{\left[u_s w_s + \frac{1}{2}w_s(v_s^2 + w_s^2)\right]\mathbf{G}_{11} + \left[u_s v_s + \frac{1}{2}v_s(v_s^2 + w_s^2)\right]\mathbf{G}_{12} + \frac{1}{2}(v_s^2 + w_s^2)\mathbf{G}_2\right\} ds,$$

$$\mathbf{G}_{11} = \{0, 0, E_{3s}, 0, -E_{5s}, 0, 0, 0, E_{4s}, 0, -E_{6s}\}^t, \tag{8}$$

$$\mathbf{G}_{12} = \{0, E_{3s}, 0, 0, 0, E_{5s}, 0, E_{4s}, 0, 0, 0, E_{6s}\}^t, \quad \mathbf{G}_2 = \{E_{1s}, 0, 0, 0, 0, 0, E_{2s}, 0, 0, 0, 0, 0\}^t.$$

We write expressions for nonzero matrix elements $\mathbf{K}_l$, $\mathbf{K}_{11}$, $\mathbf{K}_{12}$, $\mathbf{K}_{21}$, $\mathbf{K}_{22}$, and $\mathbf{K}_{23}$ (the symmetric side is omitted)

$$K_{l,11} = K_{l,77} = -K_{l,17} = d_1/L, \quad K_{l,22} = K_{l,88} = -K_{l,28} = 12d_3/L^3, \quad K_{l,44} = K_{l,10\,10} = -K_{l,4\,10} = d_4/L,$$

$$K_{l,33} = K_{l,99} = -K_{l,39} = 12d_2/L^3, \quad K_{l,55} = K_{l,11\,11} = 2K_{l,5\,11} = 4d_2/L, \quad K_{l,66} = K_{l,12\,12} = 2K_{l,6\,12} = 4d_3/L,$$

$$K_{l,35} = K_{l,3\,11} = -K_{l,59} = -K_{l,9\,11} = -6d_2/L^2, \quad K_{l,26} = K_{l,2\,12} = -K_{l,68} = -LK_{l,8\,12} = 6d_3/L^2,$$

$$d_1 = EF; \quad d_2 = EJ_1, \quad d_3 = EJ_2, \quad d_4 = GJ_k;$$

$$K_{11,13} = E_{1s}E_{3s}, \quad K_{11,15} = -E_{1s}E_{5s}, \quad K_{11,19} = E_{1s}E_{4s}, \quad K_{11,1\,11} = -E_{1s}E_{6s},$$

$$K_{11,37} = E_{2s}E_{3s}, \quad K_{11,57} = -E_{2s}E_{5s}, K_{11,79} = E_{2s}E_{4s}, \quad K_{11,7\,11} = -E_{2s}E_{6s};$$

$$K_{12,12} = E_{1s}E_{3s}, \quad K_{12,16} = -E_{1s}E_{5s}, \quad K_{12,18} = E_{1s}E_{4s}, \quad K_{12,1\,12} = E_{1s}E_{6s}, \quad K_{12,27} = E_{2s}E_{3s},$$

$$K_{12,67} = E_{2s}E_{5s}, \quad K_{12,78} = E_{2s}E_{4s}, \quad K_{12,7\,12} = E_{2s}E_{6s}, K_{21,33} = E_{3s}^2, \quad K_{21,35} = -E_{3s}E_{5s},$$

$$K_{21,39} = E_{3s}E_{4s}, \quad K_{21,3\,11} = -E_{3s}E_{6s}, \quad K_{21,55} = E_{5s}^2, \quad K_{21,59} = -E_{4s}E_{5s}, \quad K_{21,5\,11} = E_{5s}E_{6s},$$

$$K_{21,99} = E_{4s}^2, \quad K_{21,9\,11} = -E_{4s}E_{6s}, \quad K_{21,11\,11} = E_{6s}^2;$$

$$K_{22,23} = E_{3s}^2, \quad K_{22,25} = -E_{3s}E_{5s}, \quad K_{22,29} = E_{3s}E_{4s}, \quad K_{22,2\,11} = -E_{3s}E_{6s}, \quad K_{22,56} = -E_{5s}^2, \quad K_{22,58} = -E_{4s}E_{5s},$$

$$K_{22,5\,12} = -E_{5s}E_{6s}, \quad K_{22,36} = E_{3s}E_{5s}, \quad K_{22,38} = E_{3s}E_{4s}, \quad K_{22,3\,12} = E_{3s}E_{6s}, \quad K_{22,69} = E_{4s}E_{5s},$$

$$K_{22,6\,11} = -E_{5s}E_{6s}, \quad K_{22,89} = E_{4s}^2, \quad K_{22,8\,11} = -E_{4s}E_{6s}, \quad K_{22,9\,12} = E_{4s}E_{6s}, \quad K_{22,11\,12} = -E_{6s}^2;$$

$$K_{23,22} = E_{3s}^2, \quad K_{23,26} = E_{3s}E_{5s}, \quad K_{23,28} = E_{3s}E_{4s}, \quad K_{23,2\,12} = E_{3s}E_{6s}, \quad K_{23,66} = E_{5s}^2,$$

$$K_{23,68} = E_{4s}E_{5s}, \quad K_{23,6\,12} = E_{5s}E_{6s}, \quad K_{23,88} = E_{4s}^2, \quad K_{23,8\,12} = E_{4s}E_{6s}, \quad K_{23,12\,12} = E_{6s}^2.$$

Numerical Integration Over the Element. In our program the definite integrals in expressions (7) and (8) are calculated by the Gauss method using the formulas

$$\int_0^L f(s)\,ds = \frac{L}{2}\int_{-1}^{1} f\Big(\frac{L}{2}(t+1)\Big)\,dt = \frac{L}{2}\sum_{i=1}^{n} H_i f\Big(\frac{L}{2}(x_i+1)\Big).$$

In the case of 3-point integration, we have

$$\begin{aligned} H_1 &= 0.555555555555556, & x_1 &= \ \ 0.774596669241483,\\ H_2 &= 0.888888888888889, & x_2 &= \ \ 0,\\ H_3 &= 0.555555555555556, & x_3 &= -0.774596669241483, \end{aligned}$$

and for 4-point integration,

$$\begin{aligned} H_1 &= 0.347854845137454, & x_1 &= \ \ 0.861136311594053,\\ H_2 &= 0.652145154862546, & x_2 &= \ \ 0.339981043584856,\\ H_3 &= 0.652145154862546, & x_3 &= -0.339981043584856,\\ H_4 &= 0.347854845137454, & x_4 &= -0.861136311594053. \end{aligned}$$

Coordinate Transform. The general Hessian matrix and the strain-energy gradient of the structure are constructed in the usual fashion by transforming the Hessian matrix and the potential-energy gradients of individual bar elements from the local $\{x, y, z\}$ axes into the global Cartesian system $\{x', y', z'\}$ (Fig. 1) with subsequent summation. The vectors $\mathbf{G}_e$ and matrices $\mathbf{H}_e$ are transformed by the formulas

$$\mathbf{G}'_e = \mathbf{N}^t\mathbf{G}_e, \quad \mathbf{H}'_e = \mathbf{N}^t\mathbf{H}_e\mathbf{N},$$

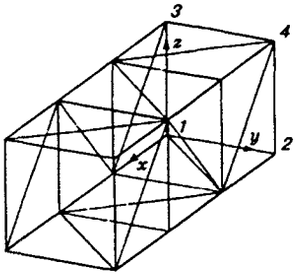

Fig. 2

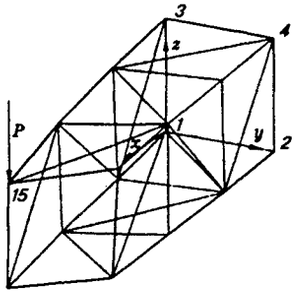

Fig. 3

where

$$\mathbf{N} = \begin{pmatrix} [l] & 0 & 0 & 0 \\ 0 & [l] & 0 & 0 \\ 0 & 0 & [l] & 0 \\ 0 & 0 & 0 & [l] \end{pmatrix}; \qquad [l] = \begin{pmatrix} l_{xx'} & l_{xy'} & l_{xz'} \\ l_{yx'} & l_{yy'} & l_{yz'} \\ l_{zx'} & l_{zy'} & l_{zz'} \end{pmatrix};$$

and x, y, and z are the coordinates of nodes 0 or 1 of the bar finite element.

The vector $\mathbf{r}_2^t = \{l_{yx'}, l_{yy'}, l_{yz'}\}$ determines the direction of the local y axis, and the vector $\mathbf{r}_3^t = \{l_{zx'}, l_{zy'}, l_{zz'}\}$ is found from the vector product $\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$, $\mathbf{r}_1^t = \{l_{xx'}, l_{xy'}, l_{xz'}\}$.

Since matrix $\mathbf{N}$ is orthogonal, the inverse transformation of the vector from the global to the local system, which is required for calculating the nonlinear vectors and matrices in the iteration procedure, is performed by the formula $\mathbf{u}_e = \mathbf{N}\mathbf{u}'_e$.

Algorithm of Investigation. The Newton–Kantorovich iterative method is described by the formulas

$$\mathbf{H}(\mathbf{u}^n)\,\delta\mathbf{u}^n = \mathbf{P} - \mathbf{G}(\mathbf{u}^n), \quad \mathbf{u}^{n+1} = \mathbf{u}^n + \delta\mathbf{u}^n.$$

As a first approximation of $\mathbf{u}^1$, we can use either the zero vector or the solution for the previous loading step. The convergence can be controlled by three parameters:

(1) the maximum relative error

$$\Delta_1 = \max_{1 \leqslant i \leqslant m} \left(\left| \frac{\delta u_i^n}{u_i^{n+1}} \right| \right),$$

where m is the dimension of the vectors $\mathbf{u}^{n+1}$ and $\mathbf{u}^n$;

(2) the maximum absolute error

$$\Delta_2 = \delta u_k^n, \quad \left|\delta u_k^n\right| = \max_{1 \leqslant i \leqslant m} \left|\delta u_i^n\right|;$$

and (3) the truncation norm

$$\Delta_3 = \sqrt{\sum_{i=1}^{n} [P_i - G_i(\mathbf{u}^n)]^2}.$$

The stability of the structure is analyzed using the energy criterion which is reduced to control of the positiveness of the elements of diagonal matrix $\mathbf{D}$ in the $\mathbf{L}^t\mathbf{D}\mathbf{L}$ expansion of the Hessian matrix $\mathbf{H}$. The appearance of at least one nonpositive element indicates that the structure is in an unstable equilibrium state.

To study stability in the initial linear state it will suffice to perform two iterations with a zero vector as a first approximation at each loading level. The appearance of negative elements in matrix $\mathbf{D}$ in the second iteration indicates the unstable equilibrium of the structure at this loading level.

Calculations. The algorithm and the program developed from it were tested in stability analysis of a hinged circular bar under axial compression. The length of the bar was 100 cm, the cross-section diameter

was 1 cm, and the material used was duralumin.

Approximating the bar by one finite element we obtain a critical load that is 25% higher than the Euler load:

$$P_c = \frac{\pi^2 EJ}{L^2},$$

where EJ is the minimum bending stiffness and L is the length of the element. Approximating by two elements, we obtain a critical load that is only 0.5% higher than the theoretical value.

The distribution of moments over bar length was tested using the problem of cantilevered bending. As an example, we also considered a spatial truss consisting of steel hollow bars of 50 mm outside and 48 mm inside diameters (Fig. 2). The truss was hinged at nodes 1–4.

Figure 3 shows the deformed state of this truss with a concentrated force applied at node 15. There is no loss of stability within the elastic range of material (for force P not greater than 150,000 N). The theoretical critical load is 550,000 N.

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